Improper Integral

1. **(a)** $\frac{\pi}{4}$ **(b)** diverges **(c)** 2 **(d)** diverges **(e)** $\frac{1}{2}$

(f) 2 **(g)** diverges **(h)** $\frac{\pi}{20}$ **(i)** π **(j)** diverges

(k) $\frac{\pi}{ab}$ **(l)** 1 **(m)** $\frac{1}{2}$ **(n)** -1 **(o)** a

(p) π **(q)** ln 2 **(r)** $\frac{3}{2}-\ln 4$

2. $I_n = \int_0^\infty x^n e^{-x}dx = -\int_0^\infty x^n d\left(e^{-x}\right) = -x^n e^{-x}\Big|_0^\infty + \int_0^\infty nx^{n-1}e^{-x}dx = -\lim_{t\to\infty}\frac{t^n}{e^t} + n\int_0^\infty x^{n-1}e^{-x}dx = nI_{n-1}$

since $\lim_{t\to\infty}\frac{t^n}{e^t} \underset{(L.H.R)}{=} \lim_{t\to\infty}\frac{nt^{n-1}}{e^t} \underset{(L.H.R)}{=} \lim_{t\to\infty}\frac{n(n-1)t^{n-2}}{e^t} = \ldots = \lim_{t\to\infty}\frac{n!}{e^t} = 0$

3. **(a)** **(Method 1 , by parametric integration)**

Let $I_n = \int_{-\infty}^\infty \frac{dx}{\left(ax^2+2bx+c\right)^n}$ $(ac - b^2 > 0)$

$$ax^2+2bx+c = a\left[x^2+2\left(\frac{b}{a}\right)x+\left(\frac{b}{a}\right)^2\right]-\frac{b^2}{a}+c = \left[\sqrt{a}\left(x+\frac{b}{a}\right)\right]^2+\left(\sqrt{\frac{ac-b^2}{a}}\right)^2$$

Put $y = \sqrt{a}\left(x+\frac{b}{a}\right)$, $\lambda = \sqrt{\frac{ac-b^2}{a}}$, $(ac - b^2 > 0)$

$$I_1 = \int_{-\infty}^\infty \frac{dx}{ax^2+2bx+c} = \frac{1}{\sqrt{a}}\int_{-\infty}^\infty \frac{dy}{y^2+\lambda^2}$$

Consider $J_n = \int_{-\infty}^\infty \frac{dy}{\left(y^2+\lambda^2\right)^n}$, $J_1 = \int_{-\infty}^\infty \frac{dy}{y^2+\lambda^2} = \left[\frac{1}{\lambda}\tan^{-1}\frac{y}{\lambda}\right]_{-\infty}^\infty = \frac{\pi}{\lambda}$ …. (1)

Differentiate (1) w.r.t λ , $\int_{-\infty}^\infty \frac{dy}{y^2+\lambda^2}(-2\lambda) = -\frac{\pi}{\lambda^2} \Rightarrow J_2 = \frac{1}{2}\frac{\pi}{\lambda^3}$ …. (2)

Differentiate (2) w.r.t. λ , $\int_{-\infty}^\infty \frac{dy}{\left(y^2+\lambda^2\right)^2}(-2)(2\lambda) = -\frac{1}{2}(-3)\frac{\pi}{\lambda^4} \Rightarrow J_3 = \frac{1}{2}\frac{3}{4}\frac{\pi}{\lambda^5}$

We claim : $J_n = \frac{1}{2}\frac{3}{4}\frac{5}{6}\ldots\frac{2n-3}{2n-2}\frac{\pi}{\lambda^{2n-1}}$ and prove by mathematical induction.

It has proved for the case n = 1. Assume $J_k = \int_{-\infty}^\infty \frac{dy}{\left(y^2+\lambda^2\right)^k} = \frac{1}{2}\frac{3}{4}\frac{5}{6}\ldots\frac{2k-3}{2k-2}\frac{\pi}{\lambda^{2k-1}}$ …. (3)

Differentiate (3) w.r.t. λ , $\int_{-\infty}^\infty \frac{dy}{\left(y^2+\lambda^2\right)^{k+1}}(-k)(2\lambda) = \frac{1}{2}\frac{3}{4}\frac{5}{6}\ldots\frac{2k-3}{2k-2}\frac{\pi}{\lambda^{2k}}(-2k-1)$

$\therefore$ $J_{k+1} = \frac{1}{2}\frac{3}{4}\frac{5}{6}\ldots\frac{2k-3}{2k-2}\frac{2k+1}{2k}\frac{\pi}{\lambda^{2k+1}}$ and the proposition is true for n = k + 1.

$$I_n = \int_{-\infty}^\infty \frac{dx}{\left(ax^2+2bx+c\right)^n} = \frac{1}{\sqrt{a}}J_n = \frac{1}{\sqrt{a}}\frac{1}{2}\frac{3}{4}\frac{5}{6}\ldots\frac{2n-3}{2n-2}\frac{\pi}{\lambda^{2n-1}} = \frac{1}{2}\frac{3}{4}\frac{5}{6}\ldots\frac{2n-3}{2n-2}\frac{\pi a^{n-1}}{\sqrt{ac-b^2}^{\,2n-1}}$$

(Method 2, by reduction formula)

$$J_{n+1} = \int_{-\infty}^{\infty} \frac{dy}{(y^2+\lambda^2)^{n+1}} = \frac{2n-1}{2n}\frac{1}{\lambda^2}J_n, \quad J_1 = \frac{\pi}{4\lambda}$$

$$J_n = \int_{-\infty}^{\infty} \frac{dy}{(y^2+\lambda^2)^n} = \int_{-\infty}^{\infty} \frac{(y^2+\lambda^2)dy}{(y^2+\lambda^2)^{n+1}} = \int_{-\infty}^{\infty} \frac{y^2dy}{(y^2+\lambda^2)^{n+1}} + \int_{-\infty}^{\infty} \frac{\lambda^2dy}{(y^2+\lambda^2)^{n+1}}$$

$$= -\frac{1}{2n}\int_{-\infty}^{\infty} yd\left[\frac{1}{(y^2+\lambda^2)^n}\right] + \lambda^2 J_{n+1} = -\frac{1}{2n}\left\{\left[\frac{y}{(y^2+\lambda^2)^n}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{dy}{(y^2+\lambda^2)^n}\right\} + \lambda^2 J_{n+1}$$

$$= \frac{1}{2n}J_n + \lambda^2 J_{n+1} \quad \Rightarrow \quad J_{n+1} = \int_{-\infty}^{\infty} \frac{dy}{(y^2+\lambda^2)^{n+1}} = \frac{2n-1}{2n}\frac{1}{\lambda^2}J_n, \quad J_1 = \frac{\pi}{4\lambda}$$

(b) $I_n = \int_1^{\infty} \frac{dx}{x(x+1)\dots(x+n)}$

$$I_{n+1} = \int_1^{\infty} \frac{dx}{x(x+1)\dots(x+n)(x+n+1)} = \frac{1}{n+1}\int_1^{\infty} \frac{(x+n+1)-x}{x(x+1)\dots(x+n)(x+n+1)}dx$$

$$\therefore \quad (n+1)I_{n+1} = I_n - \int_1^{\infty} \frac{dx}{(x+1)\dots(x+n)(x+n+1)} \qquad \dots. \quad (1)$$

Let $L = \int_1^{\infty} \frac{dx}{(x+1)\dots(x+n)(x+n+1)}$

Put $y = x + 1$, $dx = dy$. When $x = 1, y = 2$ and when $x \to \infty$, $y \to \infty$.

$$\therefore \quad L = \int_2^{\infty} \frac{dx}{y(y+1)\dots(y+n)} = \int_2^{\infty} \frac{dx}{x(x+1)\dots(x+n)} = \int_1^{\infty} \frac{dx}{x(x+1)\dots(x+n)} - \int_1^{2} \frac{dx}{x(x+1)\dots(x+n)} \quad \dots. \quad (2)$$

Subst. (2) in (1), $\quad (n+1)I_{n+1} = \int_1^2 \frac{dx}{x(x+1)\dots(x+n)} \qquad \dots. \quad (3)$

Let $J_n = \int_1^2 \frac{dx}{x(x+1)\dots(x+n)}$

$$\frac{1}{x(x+1)\dots(x+n)} = \frac{A_0}{x} + \frac{A_1}{x+1} + \dots + \frac{A_k}{x+k} + \dots + \frac{A_n}{x+n}$$

$1 = A_0(x + 1)\dots(x + n) + A_1x (x + 2) \dots(x + n) + \dots + A_kx(x + 1)\dots(x + k - 1)(x + k + 1)\dots(x + n) + \dots + A_nx(x + 1)\dots(x + n - 1)$

Put $x = -k$, $\quad 1 = (-1)^k k!(n-k)!A_k \quad \Rightarrow A_k = (-1)^k \frac{1}{k!(n-k)!}$

$$\therefore \quad \frac{1}{x(x+1)\dots(x+n)} = \sum_{k=0}^{n} (-1)^k \frac{1}{k!(n-k)!}\left(\frac{1}{x+k}\right)$$

$$J_n = \int_1^2 \frac{dx}{x(x+1)\dots(x+n)} = \int_1^2 \sum_{k=0}^{n} (-1)^k \frac{1}{k!(n-k)!}\left(\frac{1}{x+k}\right)dx = \sum_{k=0}^{n} (-1)^k \frac{1}{k!(n-k)!}\ln(x+k)\Big|_1^2$$

$$= \sum_{k=0}^{n} (-1)^k \frac{1}{k!(n-k)!}\ln\left(\frac{2+k}{1+k}\right)$$

From (3), $I_{n+1} = \frac{1}{n+1}\sum_{k=0}^{n} (-1)^k \frac{1}{k!(n-k)!}\ln\left(\frac{2+k}{1+k}\right) = \frac{1}{(n+1)!}\sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!}\ln\left(\frac{2+k}{1+k}\right)$

$$= \frac{1}{(n+1)!}\sum_{k=0}^{n} (-1)^k C_k^n \ln\left(\frac{2+k}{1+k}\right)$$

For your checking, $I_1 = \ln 2$

$$I_2 = \frac{1}{2!}(2\ln 2 - \ln 3), \qquad I_3 = \frac{1}{3!}(5\ln 2 - 3\ln 3), \qquad I_4 = \frac{1}{4!}(12\ln 2 - 6\ln 3 - \ln 5)$$

(c) $I_n = \int_0^1 \frac{x^n dx}{\sqrt{1-x^2}} = -\int_0^1 x^{n-1} d\sqrt{1-x^2} = -\left[x^{n-1}\sqrt{1-x^2}\right]_0^1 + (n-1)\int_0^1 x^{n-2}\sqrt{1-x^2}\,dx$

$$= (n-1)\int_0^1 x^{n-2}\sqrt{1-x^2}\,dx = (n-1)\int_0^1 \frac{x^{n-2}(1-x^2)}{\sqrt{1-x^2}}dx = (n-1)[I_{n-2} - I_n]$$

$$\therefore \quad I_n = \frac{n-1}{n} I_{n-2} = \frac{n-1}{n}\frac{n-3}{n-2} I_{n-4} = \frac{n-1}{n}\frac{n-3}{n-2}\frac{n-5}{n-4} I_{n-6} = \ldots$$

$$= \begin{cases} \frac{n-1}{n}\frac{n-3}{n-2}\frac{n-5}{n-4}\cdots\frac{1}{2} I_0, & n \in 2\mathbf{N} \\ \frac{n-1}{n}\frac{n-3}{n-2}\frac{n-5}{n-4}\cdots\frac{2}{3} I_1, & n \in 2\mathbf{N}+1 \end{cases} = \begin{cases} \frac{(n-1)(n-5)\ldots 3.1}{n(n-2)(n-4)\ldots 4.2}\frac{\pi}{2}, & n \in 2\mathbf{N} \\ \frac{(n-1)(n-5)\ldots 2.1}{n(n-2)(n-4)\ldots 5.3}, & n \in 2\mathbf{N}+1 \end{cases}$$

since $I_1 = \int_0^1 \frac{x dx}{\sqrt{1-x^2}} = -\frac{1}{2}\int_0^1 \frac{d(1-x^2)}{\sqrt{1-x^2}} = -\frac{1}{2}\times 2\left[\sqrt{1-x^2}\right]_0^1 = 1$

$I_0 = \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}$, by putting $x = \sin\theta$

4. **(i)** Put $y = \frac{\pi}{2} - x$, $dy = -dx$, When $x = \frac{\pi}{2}$, $y = 0$ and $x = 0$, $y = \frac{\pi}{2}$

$$I = \int_0^{\pi/2} \ln(\sin x)dx = \int_{\pi/2}^{0} \ln\sin\left(\frac{\pi}{2} - y\right)(-dy) = \int_0^{\pi/2} \ln(\cos y)dy = \int_0^{\pi/2} \ln(\cos x)dx$$

$$\therefore 2I = \int_0^{\pi/2} \ln(\sin x)dx + \int_0^{\pi/2} \ln(\cos x)dx = \int_0^{\pi/2} [\ln(\sin x) + \ln(\cos x)]dx = \int_0^{\pi/2} \ln(\sin x\cos x)dx$$

$$= \int_0^{\pi/2} \ln(\sin 2x - \ln 2)dx = \frac{1}{2}\int_0^{\pi/2} \ln(\sin 2x)dx - \frac{\pi}{4}\ln 2 \qquad \ldots. \quad (1)$$

(ii) Let $2x = y$, $2dx = dy$

$$\int_0^{\pi/2} \ln(\sin 2x)dx = \frac{1}{2}\int_0^{\pi} \ln(\sin y)dy = \frac{1}{2}\int_0^{\pi} \ln(\sin y)dy = \frac{1}{2}\int_0^{\pi/2} \ln(\sin x)dx + \frac{1}{2}\int_{\pi/2}^{\pi} \ln(\sin x)dx$$

Putting $x = \frac{\pi}{2} + z$ in the last integral on the right,

$$\int_{\pi/2}^{\pi} \ln(\sin x)dx = \int_0^{\pi/2} \ln\sin\left(\frac{\pi}{2} + z\right)dz = \int_0^{\pi/2} \ln\cos z dz = \int_0^{\pi/2} \ln\sin z dz = \int_0^{\pi/2} \ln\sin x dx$$

$$\therefore \quad \int_0^{\pi/2} \ln(\sin 2x)dx = \int_0^{\pi/2} \ln(\sin x)dx = I \qquad \ldots. \quad (2)$$

$$(2) \downarrow (1), \quad 2I = I - \frac{\pi}{4}\ln 2 \;, \quad \therefore \quad I = \int_0^{\pi/2} \ln(\sin x)dx = \frac{\pi}{2}\ln\frac{1}{2}$$

5. $I_{2n} = \int \cos^{2n} x \; dx = \int \cos^{2n-1} x \; d(\sin x) = \sin x \cos^{2n-1} x - \int \sin x d\left(\cos^{2n-1} x\right)$

$$= \sin x \cos^{2n-1} x + (2n-1)\int \sin^2 x \cos^{2n-2} x dx = \sin x \cos^{2n-1} x + (2n-1)\int \left(1 - \cos^2 x\right)\cos^{2n-2} x dx$$

$$= \sin x \cos^{2n-1} x + (2n-1)\left[I_{2n-2} - I_{2n}\right]$$

$$\therefore \quad I_{2n} = \frac{1}{2n}\sin x \cos^{2n-1} x + \frac{2n-1}{2n} I_{2n-2}$$

Let $f(x) = \cos^{2n} x$, Since $f(x + \pi) = \cos^{2n}(x + \pi) = \cos^{2n} x$, $f(x)$ is periodic with period $= \pi$.

Also, $f\left(x + \frac{\pi}{2}\right) = \cos^{2n}\left(x + \frac{\pi}{2}\right) = \cos^{2n}\left(x - \frac{\pi}{2}\right) = f\left(x - \frac{\pi}{2}\right)$, the function is symmetric about $x = \pi/2$.

$$\int_0^{r\pi/2} \cos^{2n} x \, dx = r\int_0^{\pi/2} \cos^{2n} x \, dx = rI_{2n} = r\left[\frac{1}{2n}\sin x \cos^{2n-1} x\right]_0^{\pi/2} + r\frac{2n-1}{2n} I_{2n-2} = r\frac{2n-1}{2n} I_{2n-2}$$

$$= r\frac{2n-1}{2n}\frac{2n-3}{2n-2} I_{2n-4} = \ldots = r\frac{2n-1}{2n}\frac{2n-3}{2n-2}\ldots\frac{3}{4}\frac{1}{2} I_0 = r\frac{(2n-1)(2n-3)\ldots\times 3\times 1}{(2n)(2n-2)\ldots\times 4\times 2}\frac{\pi}{2}$$

6. **(i)** $\int e^{-x^2} x^3 dx = -\frac{1}{2} e^{-x^2}\left(1 + x^2\right) + c$,

$$I_n = \int_0^x \frac{dx}{\left(1+x^2\right)^n} = \left[\frac{x}{\left(1+x^2\right)^n}\right]_0^x - \int_0^x x d\left(\frac{1}{\left(1+x^2\right)^n}\right) = \frac{x}{\left(1+x^2\right)^n} + 2n\int_0^x \frac{x^2 dx}{\left(1+x^2\right)^{n+1}}$$

$$= \frac{x}{\left(1+x^2\right)^n} + 2n\int_0^x \frac{dx}{\left(1+x^2\right)^n} - 2n\int_0^x \frac{dx}{\left(1+x^2\right)^{n+1}} = \frac{x}{\left(1+x^2\right)^n} + 2nI_n - 2nI_{n+1}$$

$$\therefore \quad I_{n+1} = \frac{x}{\left(1+x^2\right)^n} + 2n\int_0^x \frac{dx}{\left(1+x^2\right)^n} - 2n\int_0^x \frac{dx}{\left(1+x^2\right)^{n+1}} = \frac{1}{2n}\frac{x}{\left(1+x^2\right)^n} + \frac{2n-1}{2n} I_n$$

$$\int_0^x \frac{dx}{\left(1+x^2\right)^3} = \frac{5x + 3x^3 + 3\left(1+x^2\right)\tan^{-1} x}{8\left(1+x^2\right)^2}$$

(ii) Put $t = \tan\frac{\theta}{2}$, $d\theta = \frac{2dt}{1+t^2}$, $\cos\theta = \frac{1-t^2}{1+t^2}$

$$\int_0^{\pi} \frac{d\theta}{5 + 3\cos\theta} = \int_0^{\infty} \frac{1}{5 + 3\frac{1-t^2}{1+t^2}}\frac{2dt}{1+t^2} = \int_0^{\infty} \frac{dt}{4+t^2} = \frac{1}{2}\left[\tan^{-1}\frac{t}{2}\right]_0^{\infty} = \frac{\pi}{4}$$

$$\int_0^{\pi} \frac{(\cos\theta + 2\sin\theta)d\theta}{5 + 3\cos\theta} = \frac{1}{3}\int_0^{\pi} \frac{3\cos\theta + 6\sin\theta}{5 + 3\cos\theta} d\theta = \frac{1}{3}\int_0^{\pi} \frac{(5 + 3\cos\theta) - 5 + 6\sin\theta}{5 + 3\cos\theta} d\theta$$

$$=\frac{1}{3}\int_0^\pi d\theta-\frac{5}{3}\int_0^\pi \frac{1}{5+3\cos\theta}d\theta+\frac{1}{3}\int_0^\pi \frac{6\sin\theta}{5+3\cos\theta}d\theta=\frac{\pi}{3}-\frac{5}{3}\frac{\pi}{4}-\frac{2}{3}\int_0^\pi \frac{d(5+3\cos\theta)}{5+3\cos\theta}$$

$$=-\frac{\pi}{12}-\frac{2}{3}\left[\ln(5+3\cos\theta)\right]_0^\pi=-\frac{\pi}{12}-\frac{2}{3}\left[\ln 2-\ln 8\right]=-\frac{\pi}{12}+\frac{4\ln 2}{3}$$

7. **(i)** $\sqrt{3+2x-x^2}-3\sin^{-1}\left(\frac{1-x}{2}\right)+c$ **(ii)** $\ln\left(\frac{x}{1+x}\right)-\frac{\ln(1+x)}{x}+c$

8. Let $I=\int_0^a f(a-x)dx$. Put $y=a-x$, $dy=-dx$. $\therefore$ $I=\int_a^0 f(y)(-dy)=\int_0^a f(y)dy=\int_0^a f(x)dx$

$$I=\int_0^\pi \frac{xdx}{4+\sin^2 x}=\int_0^\pi \frac{(\pi-x)dx}{4+\sin^2(\pi-x)}=\pi\int_0^\pi \frac{dx}{4+\sin^2 x}-\int_0^\pi \frac{xdx}{4+\sin^2 x}$$

$$\therefore\quad 2I=\pi\int_0^\pi \frac{dx}{4+\sin^2 x}=\pi\int_0^\pi \frac{\sec^2 xdx}{4\sec^2 x+\tan^2 x}=\pi\int_0^\pi \frac{d\tan x}{4+5\tan^2 x}=\pi\int_0^\pi \frac{d\tan x}{2^2+\left(\sqrt{5}\tan x\right)^2}$$

$$\therefore\quad I=\frac{\pi}{2}\left\{\int_0^{\pi/2}\frac{d\tan x}{2^2+\left(\sqrt{5}\tan x\right)^2}+\int_{\pi/2}^{\pi}\frac{d\tan x}{2^2+\left(\sqrt{5}\tan x\right)^2}\right\}$$

$$=\frac{\pi}{2}\left\{\lim_{k\to\frac{\pi}{2}^-}\left[\frac{2}{\sqrt{5}}\tan^{-1}\frac{\sqrt{5}}{2}\tan x\right]_0^k+\lim_{k\to\frac{\pi}{2}^+}\left[\frac{2}{\sqrt{5}}\tan^{-1}\frac{\sqrt{5}}{2}\tan x\right]_k^\pi\right\}=\frac{\pi}{2}\left\{\left(\frac{2}{\sqrt{5}}\frac{\pi}{2}-0\right)+\left[0-\frac{2}{\sqrt{5}}\left(-\frac{\pi}{2}\right)\right]\right\}=\pi^2\frac{\sqrt{5}}{20}$$

9. $I_n=\int\sec^n xdx=\int\sec^{n-2}xd(\tan x)=\sec^{n-2}x\tan x-\int\tan xd(\sec^{n-2}x)$

$$=\sec^{n-2}x\tan x-\int\tan x\cdot(n-2)\sec^{n-3}x(\sec x\tan x)dx=\sec^{n-2}x\tan x-(n-2)\int(\sec^2 x-1)\cdot\sec^{n-2}xdx$$

$$=\sec^{n-2}x\tan x-(n-2)\int\tan^2 x\cdot\sec^{n-2}xdx=\sec^{n-2}x\tan x-(n-2)I_n+(n-2)I_{n-2}$$

$\therefore$ $(n-1)I_n=\sec^{n-2}\theta\tan\theta+(n-2)I_{n-2}$.

$$8\int_0^{\pi/4}\sec^5\theta\,d\theta=2\times 4I_5=2\left[\sec^3\theta\tan\theta\Big|_0^{\pi/4}+3I_3\right]=2\left[\left(\sqrt{2}\right)^3(1)-1\times 0\right]+6I_3=4\sqrt{2}+3\left[\sec\theta\tan\theta\Big|_0^{\pi/4}+I_1\right]$$

$$=4\sqrt{2}+3\left[\sqrt{2}+I_1\right]=7\sqrt{2}+3I_1\qquad\text{…. (1)}$$

$$I_1=\int_0^{\pi/4}\sec\theta\,d\theta=\int_0^{\pi/4}\frac{\sec\theta(\sec\theta+\tan\theta)}{\sec\theta+\tan\theta}d\theta=\int_0^{\pi/4}\frac{d(\sec\theta+\tan\theta)}{\sec\theta+\tan\theta}=\ln\left|\sec x+\tan x\right|\Big|_0^{\pi/4}=\ln\left(1+\sqrt{2}\right)\qquad\text{…. (2)}$$

Subst. (2) in (1), $8\int_0^{\pi/4}\sec^5\theta\,d\theta=7\sqrt{2}+3\ln(1+\sqrt{2})$

10. $I_{m,n}=\int_0^{\pi/2}\cos^m\theta\sin^n\theta\,d\theta=\frac{1}{n+1}\int_0^{\pi/2}\cos^{m-1}\theta\,d\left(\sin^{n+1}\theta\right)$

$$=\left[\frac{1}{n+1}\sin^{n+1}\theta\cos^{m-1}\theta\right]_0^{\pi/2}+\frac{m-1}{n+1}\int_0^{\pi/2}\sin^{n+1}\theta\cos^{m-2}\theta\sin\theta\,d\theta=\frac{m-1}{n+1}\int_0^{\pi/2}\cos^{m-2}\theta\sin^{n+2}\theta\,d\theta$$

$$= \frac{m-1}{n+1} \int_0^{\pi/2} \cos^{m-2}\theta \left(1-\cos^2\theta\right) \sin^n\theta \, d\theta = \frac{m-1}{n+1} I_{m-2,n} - \frac{m-1}{n+1} I_{m,n}$$

$$\therefore \quad I_{m,n} = \frac{m-1}{m+n} I_{m-2,n}$$

Let $I = \int_0^{\infty} \frac{t^2 dt}{\left(1+t^2\right)^4}$, Put $t = \tan\theta$, $dt = \sec\theta \, d\theta$.

$$I = \int_0^{\pi/2} \frac{\tan^2\theta}{\left(1+\tan^2\theta\right)^4} \sec^2\theta d\theta = \int_0^{\pi/2} \frac{\tan^2\theta}{\sec^6\theta} d\theta = \int_0^{\pi/2} \cos^4\theta \sin^2\theta d\theta = I_{4,2} = \frac{4-1}{4+2} I_{2,2} = \frac{1}{2}\left[\frac{1}{4} I_{0,2}\right]$$

$$= \frac{1}{8} \int_0^{\pi/2} \sin^2\theta d\theta = \frac{1}{16} \int_0^{\pi/2} \left(1-\cos 2\theta\right) d\theta = \frac{1}{16}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2} = \underline{\underline{\frac{\pi}{32}}}$$